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Optimal Taxation in a Stochastic Economy:

A Cobb-Douglas Example

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I. Introduction

The presence of uncertainty about the future is a pervasive fact that has made an extremely limited appearance in the analysis of optimal taxation. As a start to understanding the ways in which stochastic economies differ from those more commonly analyzed, we have calculated equilibria in a number of simple economies. Almost all of these economies are populated by individuals who maximize the expected value of a Cobb-Douglas utility function of first- and second-period consumption and labor.[1] The individual uncertainty concerns the ability to work in the second period. We consider economies with individuals who all have the same skill in the first period, who have one of two skill levels, and who are spread continuously over an interval of skills. In contrast we also examine determinate economies which have the same *ex post* possibilities.

Into these economies we introduce linear first- and second-period earnings taxes to finance a poll subsidy. We also introduce both flat- and wage-related pensions with earnings tests and calculate first-best allocations. Given the substitutability between first- and second-period labor and consumption which marks the Cobb-Douglas utility function, we find the risks are not of very great importance in the economy. (This is highlighted by consideration of a fixed coefficients example which

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[1] If we had only a single period, the model would be equivalent to a many-person certainty economy.

is the same as the Cobb-Douglas example under certainty.) Because of moral hazard (labor disincentive) problems, linear taxation is of limited value in providing insurance. In economies where workers have the same ex ante skills, nonlinear taxation (as with a pension plan plus retirement test) does noticeably better.

The presence of a range of ex ante skills in the economy introduces the need to redistribute as well as to provide insurance. The relative importance of these depends on the range of skill variation relative to the range of uncertainty about the length of working life. When we consider the case in which the range of skills is much larger (on the grounds that with very short working lives disability programs generally attempt to verify incapacity rather than simply to pay benefits to all nonworkers), the need for redistribution becomes the dominant factor in the design of the tax system. The value of adding a pension system to an economy with optimal linear taxation depends on the extent to which adjustment of the linear tax system is permitted for those potentially eligible for benefits under the public pension plan. Reducing earnings taxation enhances the ability of a pension plan to increase social welfare.

II. Consumer Choice

When an individual is subjected to uncertainty regarding the length of his working life, he would like to insure this risk (assuming that he is risk averse). If private companies do not offer this insurance, the government can provide a form of insurance through the tax and social insurance systems. In an economy of individuals who are (*ex ante*) identical, the government can tax future income (which is subject to random variations for a given individual) and give each individual the expected value of his tax payments (provided, as we assume, that there is no aggregate risk). In an economy of individuals who differ *ex ante* as well as *ex post*, a government tax scheme can both redistribute income and cushion uncertainty. To analyze these two pieces separately, we start by asking what size insurance policy the individual would wish to purchase or, equivalently, what tax rate would maximize expected utility in an economy of identical consumers.

The consumer problem is described as the choice of first- and second-period consumption and labor, with uncertainty about length of working life modeled in the following way: in the first period the individual has a marginal product n for each unit of labor worked. We assume that with probability 1-p the worker will have the same marginal product in the second period, but that with probability p the consumer will be unable to work at all. Although second-period decisions are allowed to be contingent on whether the consumer is able to work in that period, first-period decisions must be made under uncertainty.

In the current paper we assume that individuals make choices consistent with the correct optimization of their lifetime utility functions. Nonetheless, it must be recognized that individual myopia, misperceptions, or poor planning constitute important justifications for social insurance programs. A subsequent paper will deal with this issue explicitly.

We assume that no direct attempt is made to avoid the moral hazard problem of individuals having zero earnings because they choose not to work rather than having no earning ability. In particular, we allow the government to observe total earnings, but not wage rates or hours worked,[1] and no test is made of disability as a condition for providing benefits.[2] Thus the tax on future earnings provides a disincentive to work in addition to providing insurance against variability in skill. Since future labor supply decisions are being distorted, optimal tax policy will, in general, require taxing first- as well as second-period labor income.[3]

We take an individual's wage rate and the return to savings to be fixed in terms of the consumption good. We also assume that work and consumption choices in the first period in no way affect the marginal product of labor in the second period or the probability distribution of the length of working life. Thus, specifically, we exclude both the possibility of investment in human capital and occupational hazards which might be associated with first-period work. The two periods of life are not unrelated, however. An individual's savings is a carry-over from the first to the second period. Also, since a social insurance system differs from annual income taxation by relating benefits to past earnings, an

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[1]Cf. Kesselman (1976).

[2]This form of model more closely resembles retirement than disability where governments attempt to have medical examinations.

[3]Taxation of savings will also be generally desirable. (See Diamond and Mirrlees (1971).) We do not pursue this line of inquiry, examining only labor income taxation.

individual's earnings history will be a carry-over when we consider a model of social insurance. Below, we will also analyze a case where preferences are not additively separable over time. This introduces a further element of interdependence between decisions in the two periods.

Individuals are characterized by two parameters -- the marginal product of labor ("skill"), which is n in period one,[4] and the probability, 1-p, that the worker will have the same marginal product in the second period. The only alternative we allow in period two is a zero marginal product. Let us denote by x_1, x_{2L}, and x_{2U} the individual's level of consumption in the first period, in the second period if he is lucky (i.e., has the same skill), and in the second period if he is unlucky (i.e., has no earning ability). We denote by y_1 and y_{2L} the amount of work done in the first period and in the second period if he is lucky. Work is measured as the ratio of hours worked to total hours available in the relevant period; y_1 and y_{2L} are thus bounded between zero and one. Wages are normalized such that earnings are ny_1 and ny_{2L} in the two periods, and consumption is measured in terms of consumption expenditures.

In the simulation results presented below, we begin by adopting a particular form of the Cobb-Douglas utility function, so that expected utility can be written as[5]

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[4]Stern (1976, p. 127) has noted that in the standard one-period non-stochastic model the assumption of a fixed wage does not necessarily imply that the economy is characterized by constant returns to scale. Rather, the wage rate can be regarded as the marginal product of labor in the neighborhood of the optimum, with the government receiving any profits as lump-sum income. However, in our model the level of production will differ between periods, and we will make comparisons with alternative economies with different production levels altogether. Hence, the production technology in our model must be regarded as exhibiting fixed wages.

[5]Note that we have implicitly made the assumption that the disutility of the health or other problem which is responsible for cutting short the individual's working life is additively separable from the remainder of the utility function and can thus be ignored in the optimisation.

$$u = \ln(x_1) + \ln(1-y_1) + (1-p)\ln(x_{2L}) + p\ln(x_{2U}) + (1-p)\ln(1-y_{2L}). \quad (1)$$

A Cobb-Douglas formulation was used by Fair (1971) and for the simulation experiments in Mirrlees (1971). However, the restrictiveness of this formulation is well known; in particular, a unitary elasticity of substitution between labor and leisure is implausibly high. Stern (1976), for example, estimates this elasticity to be on the order of 0.4. Moreover, only with an elasticity of substitution which is less than one can a backward-bending supply curve for labor be observed. Kesselman (1976) and Feldstein (1973), as well as Stern, use constant elasticity of substitution (CES) utility functions; in general, the resultant optimal tax rates are quite sensitive to the elasticity of substitution. For the case at hand, however, we do not generalize to the CES utility function, largely for computational reasons.[1]

The consumer faces linear income taxation in each period and a poll subsidy. Since there is a perfect capital market, the timing of the payment of the subsidy is not essential. We model the subsidy as being paid in the first period. He is thus subject to two budget constraints, one for each state of nature:

$$\begin{aligned} x_{2U} &= I(s + (1-t_1)ny_1 - x_1) \\ x_{2L} &= I(s + (1-t_1)ny_1 - x_1) + (1-t_2)ny_{2L} \\ &= x_{2U} + (1-t_2)ny_{2L}. \end{aligned} \quad (2)$$

In these equations s is the poll subsidy which each individual receives in period one, t_i is the rate of proportional income taxation in period i,

[1]For computations of optimal social security without uncertainty, see Sheshinski (1977).

and I is one plus the interest rate which the consumer faces. Consumption when the consumer is unlucky equals the poll subsidy plus compounded savings. Consumption when the consumer is lucky exceeds consumption when he is unlucky by net second-period earnings. Demand equations are derived from the maximization of (1) subject to (2).

III. Stochastic Economy with Identical Individuals

We first consider an economy of identical consumers, in which societal well-being can be measured by the expected utility of a representative consumer. Rather than using expected utility as the objective function, however, we report on a monotone transform of expected utility, $w = \exp(\frac{1}{2}E(u))$. In this way, social welfare is measured as the level of consumption which would give the same expected utility, provided that there were no work and equal consumption in each period.[1] Assuming that the risks to which the consumers are subjected are uncorrelated (and that we have a continuum of consumers) so that we have a societal realization of the probability distribution, the government chooses tax rates and the poll subsidy to maximize social welfare (the representative consumer's transformed expected utility function) subject to consumer demand equations and the budget (or, equivalently, resource) constraint

$$Is = It_1y_1 + (1-p)t_2y_{2L} + IG, \qquad (3)$$

where G is the lump sum grant provided from outside the system or (if negative) the government revenue requirements. In this economy, maximizing welfare is equivalent to asking what linear taxes a single consumer would subject himself to. When G is zero the optimal tax rates are independent of n.[2] Thus the desired level of insurance is independent of skill when each consumer is on a breakeven basis.

An iterative search procedure is employed to calculate the optimal values for s, t_1, and t_2. Specifically, we conduct a grid search in

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[1]This measurement of welfare parallels that of Atkinson (1970).

[2]Individual labor supply functions are homogeneous of degree zero in n and s and demand functions are homogeneous of degree one, as is the exponentiated expected utility function. Net government expenditures are homogeneous of degree one in n, G, and s. Thus proportional changes in n and G induce a proportional change in optimal s and no change in optimal t.

(t_1,t_2) space. For each gridpoint it is necessary to calculate the optimal subsidy level, subject of course to the government budget constraint. Fortunately, since utility is increasing in s for fixed tax rates and, with positive tax rates, tax revenue is decreasing in s (leisure being a superior good), there is a unique (optimal) subsidy which balances the government budget. This is illustrated in Figure 1 for a particular set of parameter values with equal tax rates in the two periods. The frontier of the feasible set gives the optimal subsidy given the tax rates, and the point of tangency of the consumer's indifference curve and the frontier fixes the optimal tax rates.

Figure 2 is a sketch of a representative consumer's indifference curves in (t_1,t_2) space when $p = 1/3$, $n = .5$, $G = 0$, and $I = 1.25$.[3] Calculating the best tax rate to two decimal places, the optimum occurs at $t_1 = .04$, $t_2 = .17$.[4] Thus, just as low tax rates have been found to be desirable in the Cobb-Douglas case for a one-period economy with optimal redistribution, we have found that low rates are optimal when the goal is to provide pure insurance in our stochastic economy. Note that social welfare as a function of the tax rates is quite flat in the vicinity of the optimum. In fact, the introduction of optimal linear taxes increases social welfare by only 0.6 percent, and raising the taxes to twice their optimal levels results in a level of social welfare just 1.2 percent below the optimum. Social welfare does, however, fall off rather sharply

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[3]To facilitate comparison of results presented throughout this paper for various economies, we present results principally for economies where the mean skill level is .5, $p = 1/3$, and $I = 1.25$ for a government which has no outside revenue requirements or sources.

[4]At lower interest rates taxes tend to be relatively higher in period two. The ratio of output in period two to output in period one increases, but total output and the qualitative nature of the results remain essentially unchanged.

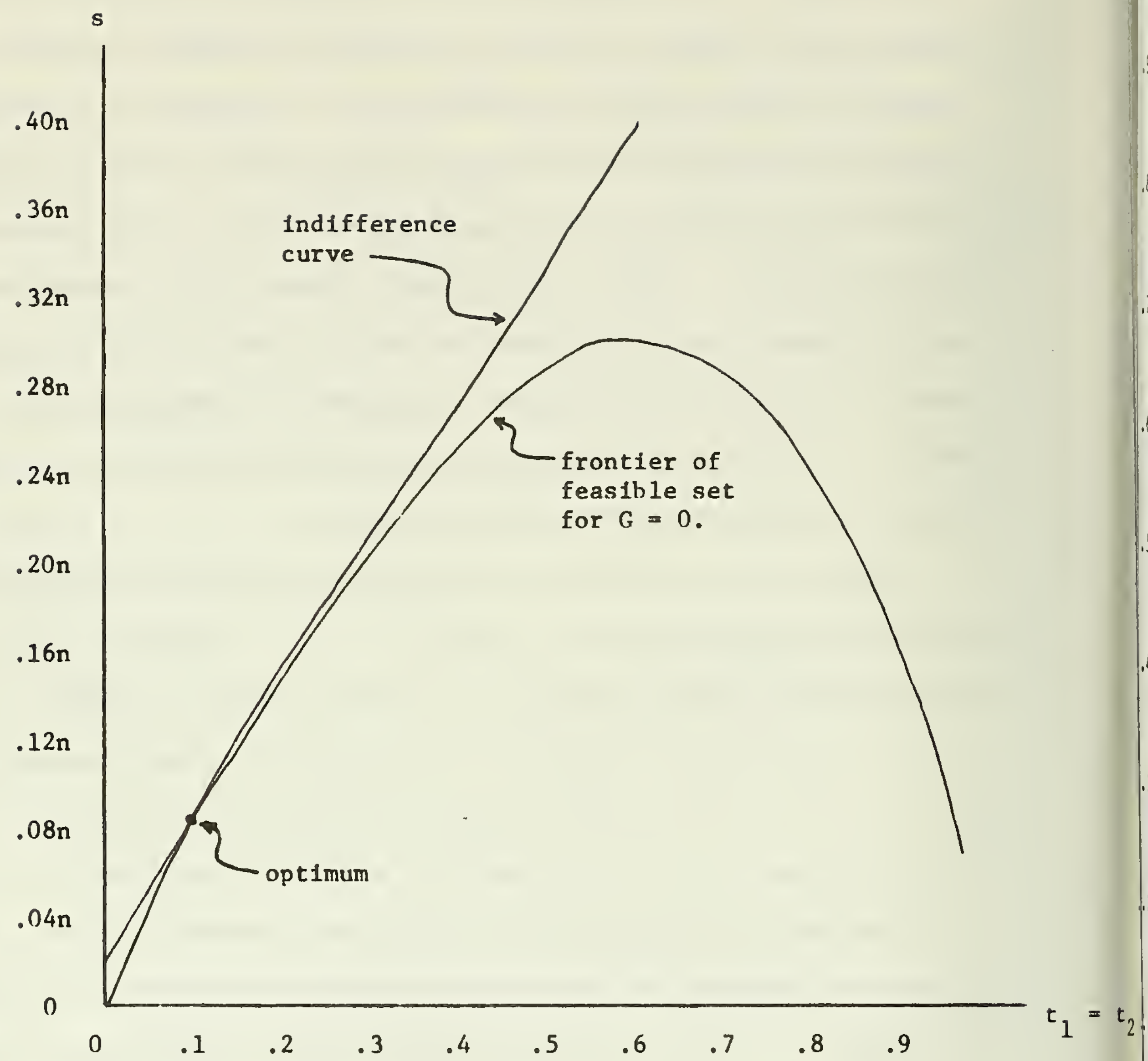

Figure 1. Consumer indifference curves and feasible tax/subsidy combinations for a stochastic economy of identical individuals with $t_1 = t_2$, $p = 1/3$, $I = 1.25$.

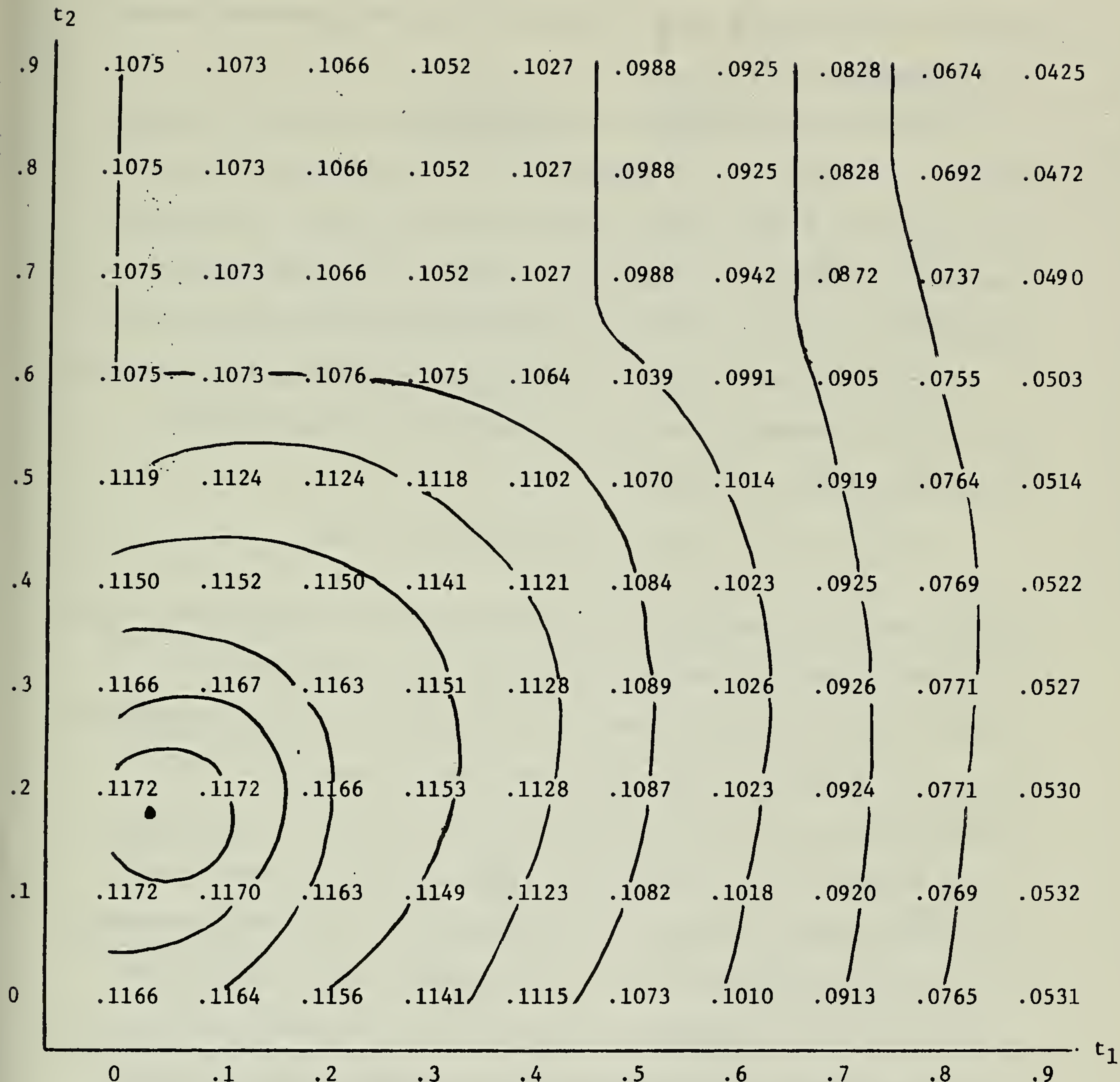

Figure 2. Social indifference curves in (t_1,t_2) space for an economy of identical consumers. Numbers given are social welfare levels for n = .5, p = 1/3, G = 0.

for tax rates in excess of those which maximize gross government revenue, as is suggested by Figure 1.

The fact that the optimum occurs where tax rates (and so deadweight burdens) are still quite low suggests that the moral hazard problem makes the insurance gain from linear taxation relatively small. To demonstrate that this is indeed so, we consider the analogous first-best maximization problem where the government can distinguish those who are able to work.[5]

In the first-best economy we can consider the government as selecting the levels of consumption and labor to maximize social welfare subject only to the resource constraint

$$Ix_1 + (1-p)x_{2L} + px_{2U} = Iny_1 + (1-p)ny_{2L} + IG. \qquad (4)$$

(This constraint is equivalent to the budget constraint, (3).) The first-best optimum, the second-best optimum, and the solution to the consumer problem without government intervention are presented in Table 1. Note that optimal linear taxation captures only 13.9 percent of the potential gains from perfect insurance. From the method of measuring social welfare, we can interpret this figure in terms of economies with equally distributed consumption. We have also calculated another measure by dividing the welfare difference by the marginal utility of resources in the hands of the government at the point of no government intervention.[6] We can then compare the additional resources needed to achieve the same welfare gain with the present discounted value of output in the economy.

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[5]Below we will consider piecewise-linear taxation, with the moral hazard problem still present.

[6]This Lagrangian is relatively constant over the range of budget differences we are considering.

	No Government Intervention	Optimal Linear Taxes	First-best Optimum
x_1	.189	.186	.209
x_{2L}	.326	.291	.261
x_{2U}	.152	.167	.261
y_1	.622	.612	.582
y_{2L}	.348	.299	.477
Output per capita in period one	.311	.306	.291
Output per capita in period two	.116	.100	.159
Present value of total output	.404	.386	.418
Aggregate first-period savings per capita	.122	.120	.082
w	.1167	.1174	.1218

Table 1. Description of the stochastic economy of identical individuals with $p = 1/3$, $n = .5$, $I = 1.25$, and $G = 0$.

In the case at hand, the benefits of perfect insurance are equal to the increase in welfare that would result from an increase in the government's budget (G) amounting to 3.5 percent of total output, while the gains available in the second-best economy are equivalent to a 0.49 percent increase in output.

It is interesting to compare resource allocations in the three economies. Introducing perfect insurance decreases both first-period output and aggregate savings. The latter decrease is most striking, with aggregate savings falling by one-third when first-best insurance is introduced into the economy with no government intervention. In the second period the first-best optimum has more work than in the absence of insurance, but the second-best economy has less work. The great fall in savings in the first-best economy (compared to no intervention) makes work more valuable and sharply increases work done in the second period. stated alternatively, without government intervention insurance is provided by large savings. When the lucky state occurs, the high level of accumulated wealth yields a smaller incentive for work. In the second-best economy, savings fall only slightly relative to no intervention and labor supply is discouraged by the tax rate. Thus output falls, rather than rises, in the second period.

IV. Determinate Economy with Identically Skilled Individuals

We have examined the taxes to which a single individual would choose to subject himself. To highlight the effects of the stochastic nature of the length of working life, we now analyze an economy which has the same *ex post* structure as the stochastic economy, but in which every individual knows whether he will be lucky or unlucky. That is, all individuals have n equal to .5 but 1/3 have zero marginal product in the second period and 2/3 have the same wage as in period one. The lucky consumers thus maximize

$$u_L = \ln(x_{1L}) + \ln(1-y_{1L}) + \ln(x_{2L}) + \ln(1-y_{2L}) \tag{5}$$

subject to the single budget constraint

$$x_{2L} = I(s + (1-t_1)ny_1 - x_{1L}) + (1-t_2)ny_{2L} \tag{6}$$

while the unlucky consumers maximize

$$u_U = \ln(x_{1U}) + \ln(1-y_{1U}) + \ln(x_{2U}) \tag{7}$$

subject to

$$x_{2U} = I(s + (1-t_1)ny_{1U} - x_{1U}) \tag{8}$$

If the fraction p of the consumers are unlucky, the social choice problem is to maximize the social welfare function

$$w = \exp\left(\tfrac{1}{2}((1-p)u_L + pu_U)\right) \tag{9}$$

subject to the budget constraint

$$s = It_1((1-p)y_{1L} + py_{1U}) + t_2(1-p)y_{2L} + IG. \tag{10}$$

Given the additive structure of preferences, the first-best optimum for this determinate economy is the same as that in the stochastic economy considered above.[7] The second-best economies are different, however. The average of the first-period labor supply functions of the lucky and the unlucky does not equal the labor supply of the individual subject to uncertainty. Similarly, average savings decisions are different. As a consequence of savings differences between the determinate lucky and those in the stochastic economy who turn out to be lucky, labor supply functions in the second period are also different. Thus the constraints imposed on a planner by the use of markets and the freedom of individual choice are different.

In the absence of government intervention it is clear that an economy with the stochastic structure we have assumed cannot have a higher level of social welfare than the corresponding determinate economy. It is interesting to note that this need not be the case at given tax rates when the government is intervening.[8] Because of the presence of uncertainty and risk aversion, a given (t_1,t_2) tax pair may raise more revenue (i.e., allow a greater lump-sum subsidy) in the stochastic economy than in the analogous determinate economy. Since risk aversion leads the stochastic economy to have generally higher (lower) output levels in the first (second) period, this will generally happen when t_1 is large relative to t_2, and not in the neighborhood of the optimum. However, within the region where this does happen there may be tax combinations

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[7]The stochastic problem requires $x_{1L} = x_{1U}$ and $y_{1L} = y_{1U}$ and perfect insurance/redistribution satisfies these constraints.

[8]This corresponds to the potential for welfare gain when the government introduces uncertainty in Stiglitz (1976) and Weiss (1976).

for which the stochastic economy will have higher social welfare than will its determinate counterpart, as can be seen from Figure 3. In our calculations, the determinate economy has a higher level of second-best optimal welfare.

Comparing stochastic and determinate economies, we will consider, in turn, differences in optimal tax rates, consumer behavior, and social welfare. In contrast to the stochastic economy, which has optimal taxes of 4 and 17 percent for our illustrative parameter set, the determinate economy has a tax of 8 percent in period two and no tax in period one. The absence of uncertainty leaves social welfare even less sensitive to changes in the tax rates: lowering the taxes to zero or doubling them results in a loss in social welfare which amounts to only 0.14 percent. And even raising the taxes to $t_1 = t_2 = 0.2$ brings about a welfare loss of less than 1 percent. More revealing is a comparison of individual behavior in the two economies.

Table 1, which summarizes behavior in the stochastic economy, can be compared directly to Table 2, which describes the corresponding determinate economy. In the absence of government intervention, the unlucky in the determinate economy work more in the first period (.667) than do the lucky (.550), due to differing anticipations regarding second-period earning ability. In the stochastic economy anticipations and therefore first-period labor supply are necessarily the same for all individuals; however, first-period production is greater in the presence of uncertainty. Similarly, the lucky consume more in the first period (.225) than do the unlucky (.167) in the determinate economy, but average consumption is smaller in the stochastic economy. After the introduction of optimal linear taxes, the effect of income variability is lessened in both economies.

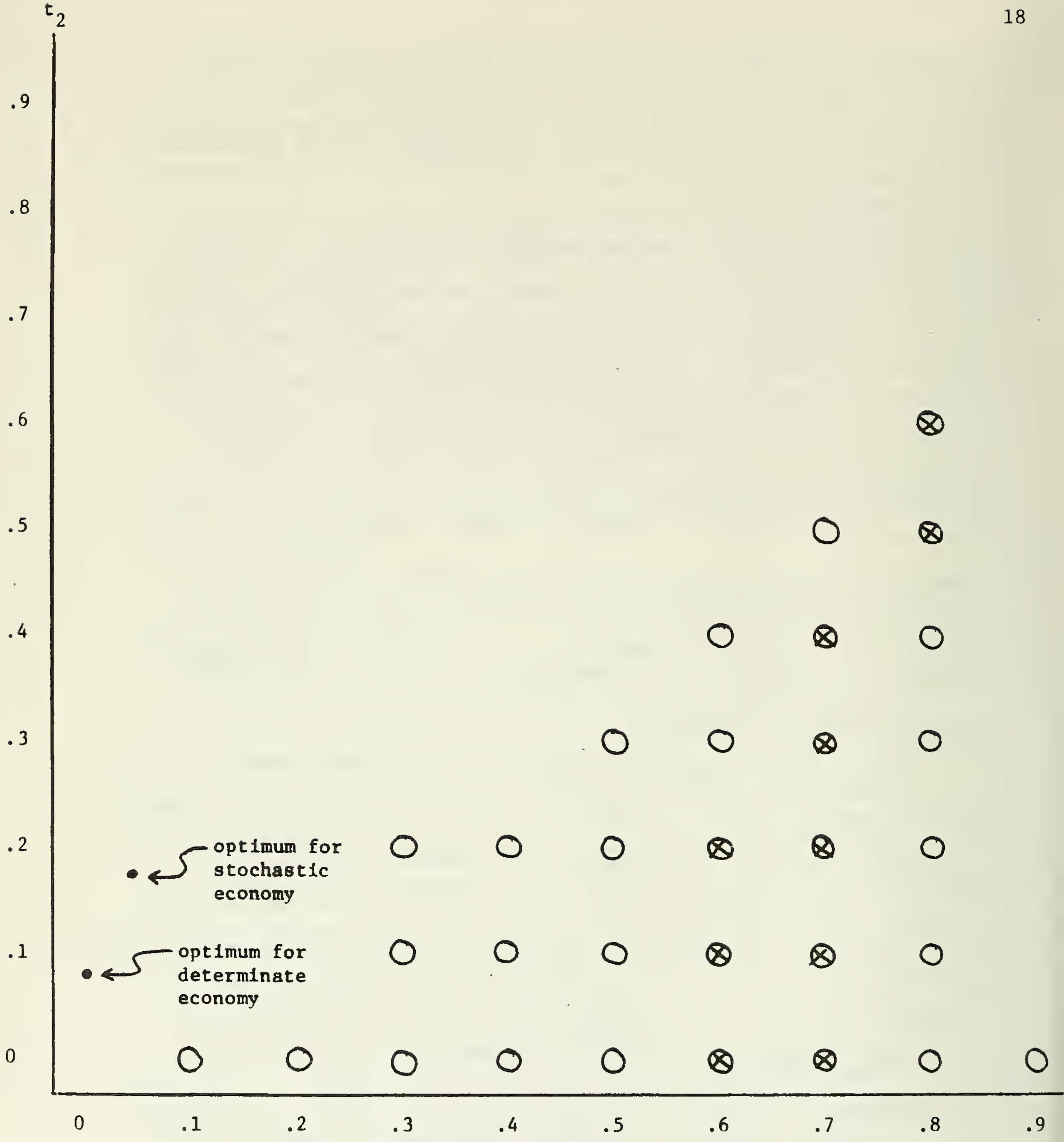

Figure 3. Comparison of the stochastic economy with n = .5, p = 1/3, I = 1.25, and G = 0 with the analogous determinate economy.

○ - grid point where tax revenue raised in the stochastic economy exceeds that raised in the determinate economy.

⊗ - grid point where the stochastic economy has a higher level of social welfare than does the determinate economy.

	No Government Intervention	Optimal Linear Taxes	First-best Optimum
x_{1L}	.225	.219	.209
x_{1U}	.167	.170	.209
x_{2L}	.281	.274	.261
x_{2U}	.208	.212	.261
y_{1L}	.550	.562	.582
y_{1U}	.667	.661	.582
y_{2L}	.438	.404	.477
Output per capita in period one	.294	.297	.291
Output per capita in period two	.146	.135	.159
Present value of total output	.411	.405	.418
Aggregate first-period savings per capita	.089	.095	.082
w	.1199	.1200	.1218

Table 2. Description of the determinate economy of individuals with skill level n = .5, 2/3 of whom are lucky and 1/3 of whom are unlucky, with I = 1.25 and G = 0.

These differences in first-period labor supply and savings functions, which are attributable to uncertainty, result in the creation of different implicit maximization problems for consumers in the second period. The second period presents a simple one-period, two-commodity maximization problem with a certain tax rate t_2 and a lump-sum subsidy equal to first-period savings, multiplied by the interest factor. Without government taxation, the individual who has been subjected to uncertainty has saved so much (.122) to reduce his risk that he chooses to work only .347, while the (lucky) man who has not been subjected to risk has saved much less (.050) in anticipation of working .437 in the second period. Similarly, a comparison of the lucky in the two economies reveals that the man subject to risk has lower first-period and higher second-period consumption, while for the unlucky man the reverse is true. The same general relationships hold after the introduction of the linear income tax. But, in addition, the disincentives of taxation result in a decrease in total output (especially in the stochastic economy where output declines by 4.4 percent) and a shift from second-period production to first-period production. This shift in production results in an increase in aggregate savings from the introduction of optimal linear taxes. In contrast, the first-best economy has less savings, having more production in the second period and less in the first.

These differences in behavior which we have noted result in smaller differences in social welfare. The magnitude of the difference is partially a consequence of the Cobb-Douglas utility function which allows a great deal of substitution. Nonetheless, welfare comparisons between the two economies do show noticeable differences from the presence of uncertainty in the economy. Without government intervention, social welfare falls only 1.5 percent short of the first-best optimum in the determinate economy -- a gap equivalent to 1.3 percent of total output.[9] In the stochastic economy the gap is almost three times as large. And while linear taxation is relatively ineffectual in curtailing the welfare loss in the stochastic economy -- 14 percent of the potential (first-best) gains are actually realized given linear taxation, it leads to even less improvement -- 8 percent of a much lower potential gain -- in the determinate economy. Thus, insuring the length of working life and redistributing between individuals with different working lives have noticeable differences.

An alternative way of thinking about the differences between stochastic and determinate economies, as described in Tables 1 and 2, is to consider changes in information, holding constant the extent of government intervention. Thus, when there is no government intervention, an innovation which permitted the lucky and the unlucky to identify themselves before making first-period decisions would increase social welfare by 2.7 percent, which is 63 percent of the total improvement from introducing information and the first-best allocation. This welfare increase is accompanied by a 1.7 percent increase in the present discounted value of output. With output going down in the first period and up in the second period, aggregate savings fall by 27 percent.

[9] As noted above, the comparison with total output comes from dividing the welfare difference by the marginal value of resources to the economy.

V. Stochastic Economy with Diverse Skills

We proceed now to a discussion of an economy in which there is uncertainty about the length of working life and different consumers have different first-period skill levels. Thus we introduce distributional as well as insurance considerations into the model. In this section and the next we will consider economies with just two skill classes. Then we will introduce a continuum of skills to examine cross-section patterns. In both cases we will consider economies where the mean skill level is .5.

Consumer demand functions are determined as in the stochastic economy specified above, and the social choice problem is one of choosing tax rates and the subsidy to maximize social welfare,

$$w = \exp(\tfrac{1}{2}\int v(n;\ t_1,t_2,s)\,dF(n)) \tag{11}$$

subject to the budget constraint

$$\begin{aligned} Is = {} & It_1\int y_1(n;\ t_1,t_2,s)\,dF(n) + \\ & (1-p)t_2\int y_{2L}(n;\ t_1,t_2,s)\,dF(n) + IG \end{aligned} \tag{12}$$

where F(n) is a distribution function giving the fraction of consumers in the economy with skill levels no higher than n, $v(n;\ t_1,t_2,s)$ is the maximum expected utility which a consumer with skill n can attain when taxes and the subsidy are as indicated, G is the per capita grant to the system, and y_1 and y_{2L} are the labor supply functions. As in the case of identical consumers, scale changes in n and G do not affect the optimal tax rates. Since the lucky workers have approximately twice the potential income of the unlucky, we start with the case where the highly skilled have about twice the marginal product of the lesser skilled. Very short working lives tend to be covered by disability insurance which attempts to evaluate loss of earning ability and so mitigate the moral hazard problem. Thus the model with a diversity of skills may be more representative of public tax transfer programs which do not attempt to measure ability. Consider the economy where half the population has skill .607, while the other half has skill .393. This choice of parameters at which to begin analysis was made to balance the importance of redistribution and insurance, in a sense to be described below. We shall also examine optimal taxes as we vary the skills, preserving the mean skill level. In Table 3 we describe the economy in the absence of government intervention. Everyone works the same amount in the first period, with the more highly skilled having proportionately higher consumption than the lesser skilled. The differences between the lucky and the unlucky are the same as those in the one-consumer stochastic economy considered above.

The introduction of linear taxation gives the social welfare grid shown in Figure 4. The optimal taxes of .13 and .20 in the two periods can be compared with the taxes an individual would choose for himself of .04 and .17. Thus the need for redistribution as well as insurance raises the desirable tax level. To get one sense of the relative importance of insurance provision and redistribution in the 1.1 percent increase in social welfare, we can examine the economy in which each skill class is taxed separately at the same tax rates, so that each individual receives as a lump-sum subsidy the expected value of his tax payments. In Table 4 we give the characteristics of the second-best economies in contrast to those of an economy with no government intervention and the first-best optima. Provision of insurance alone yields 54 percent of the social gain from the provision of both insurance and redistribution by optimal linear taxes.

	Low-skilled Unlucky	Low-skilled Lucky	High-skilled Unlucky	High-skilled Lucky	Average
			No Government Intervention		
x_1	.149	.149	.230	.230	.189
y_1	.622	.622	.622	.622	.622
x_2	.120	.256	.185	.396	.268
y_2	.0	.348	.0	.348	.232
E(u)	-4.997	-4.667	-4.130	-3.797	
$\exp(\frac{1}{2}E(u))$	.0822	.0970	.1268	.1498	w = .1139
			Optimal Linear Taxation		
x_1	.145	.145	.215	.215	.180
y_1	.575	.575	.593	.593	.585
x_2	.132	.223	.190	.338	.241
y_2	.0	.290	.0	.303	.198
E(u)	-4.812	-4.630	-4.097	-3.882	
$\exp(\frac{1}{2}E(u))$	.0902	.0988	.1289	.1436	w = .1152
			First-best Optimum		
x_1	.209	.209	.209	.209	.209
y_1	.468	.468	.656	.656	.562
x_2	.261	.261	.261	.261	.261
y_2	.0	.336	.0	.570	.452
E(u)	-3.540	-3.949	-3.976	-4.820	
$\exp(\frac{1}{2}E(u))$	.1703	.1388	.1370	.0898	w = .1242

Table 3. Description of the stochastic economy in which half of the individuals have n = .393 and the other half have n = .607, with p = 1/3, I = 1.25, and G = 0.

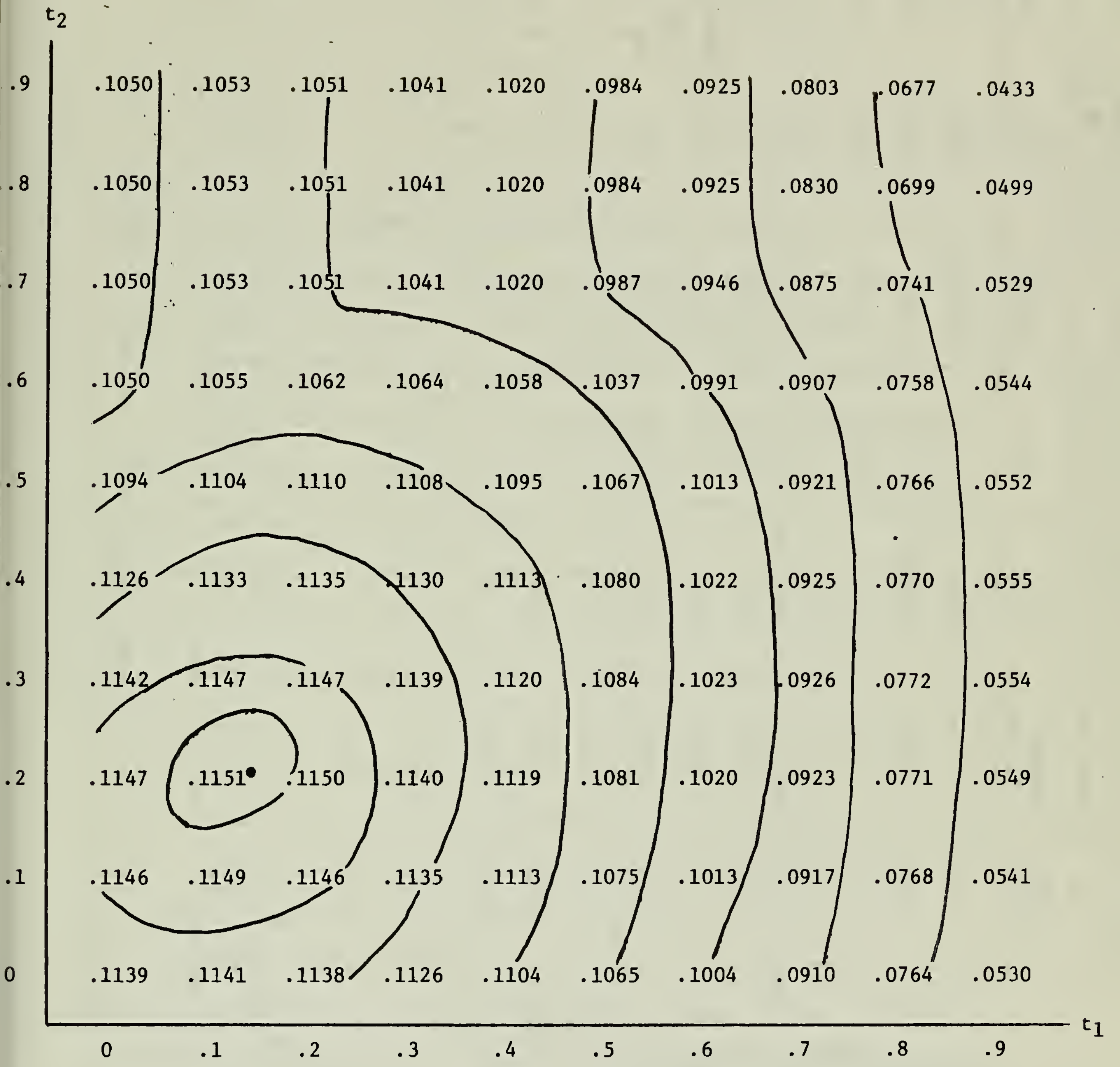

Figure 4. Social indifference curves in (t_1,t_2) space for the stochastic economy in which half of the individuals have n = .393 and the other half have n = .607. Numbers given are social welfare levels for p = 1/3, I = 1.25, G = 0.

	No Inter-vention	Second-best (Linear Taxation) Optima			First-best Optima		
		Redis-tribution	In-surance	Full	Redis-tribution	In-surance	Full
average x_{1L}	(.189)	(.183)	(.186)	(.180)	.189	(.209)	.209
average x_{1U}	(.189)	(.183)	(.186)	(.180)	.189	(.209)	.209
average x_{2L}	(.326)	(.322)	(.291)	(.281)	.326	(.261)	.261
average x_{2U}	(.152)	(.145)	(.167)	(.161)	.152	(.261)	.261
average y_{1L}	.662	(.597)	.612	(.584)	(.604)	.582	(.562)
average y_{1U}	.622	(.597)	.612	(.584)	(.604)	.582	(.562)
average y_{2L}	.348	(.355)	.299	(.297)	(.316)	.477	(.452)
output in period 1 per capita	.311	.299	.306	.293	.311	.291	.291
output in period 2 per capita	.116	.118	.100	.099	.173	.159	.239
present value of total output per capita	.404	.394	.386	.373	.450	.418	.482
(t_1,t_2)	(0,0)	(.09,0)	(.04,.17)	(.13,.20)			
present value of income tax receipts per capita	0	.027	.026	.054			
aggregate savings per capita	.122	.116	.120	.113	.122	.082	.082
w	.1139	.1142	.1146	.1152	.1190	.1190	.1242

numbers in (parentheses) are averages; others are the same for all skill levels.

Table 4. Description of the stochastic economy in which half of the individuals have n = .393 and the other half have n = .607, with p = 1/3, I = 1.25, G = 0

We can approach the question of the relative importance of insurance and redistribution in a different way by considering redistribution which does not insure length of working life. One way to do this is to tax income only in the first period. In the stochastic economy lucky and unlucky individuals both work the same amount in the first period. Thus the tax (and transfer) redistribute between skill classes but fall equally on the lucky and on the unlucky. The optimal tax rate on first-period income is 9 percent. (The economy is described in Table 4.) This yields only 23 percent of the potential gain from linear taxation.[1]

We can consider a similar division of the gains of moving to the first-best economy. Providing insurance without redistribution can be accomplished by optimizing separately in economies with just the higher skilled and just the lower skilled, redistributing optimally between the lucky and the unlucky. We can redistribute without insuring by having an optimal lump-sum transfer between a higher skilled and lower skilled person and having no other intervention. This economy was selected to make each of these moves equally important. It is interesting to note that with linear taxation, the bulk of the savings decrease comes from redistribution. With first-best tax instruments, the bulk of the much larger decline in savings comes from the provision of insurance.

In Table 5 we relate optimal taxes to the skill mix in the economy. The optimal tax rates are steady where it is optimal to have the lower skilled individual not work. This occurs when the ratio of the higher skill level to the lower skill level exceeds approximately 4:1.

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[1]If we optimized second-period taxation, with the first-period tax set equal to zero, social welfare would be .1148 and the tax rate would be .8 percent.

	Stochastic Economy				Determinate Economy			
	no intervention	optimal linear taxes			no intervention	optimal linear taxes		
Skill Levels	w	w	t_1	t_2	w	w	t_1	t_2
.5, .5	.1167	.1174	.04	.17	.1199	.1200	.00	.08
.45, .55	.1161	.1169	.06	.17	.1193	.1195	.02	.09
.40, .60	.1143	.1155	.12	.20	.1175	.1179	.08	.13
.35, .65	.1113	.1135	.19	.23	.1144	.1157	.16	.18
.30, .70	.1069	.1112	.27	.27	.1099	.1131	.25	.24
.25, .75	.1010	.1090	.34	.31	.1038	.1107	.33	.31
.20, .80	.0933	.1075	.41	.37	.0959	.1090	.40	.37
.15, .85	.0833	.1079	.56	.50	.0856	.1094	.56	.52
.10, .90	.0700	.1143	.56	.50	.0719	.1158	.56	.52
.05, .95	.0508	.1206	.56	.50	.0523	.1223	.56	.52

Table 5. Social welfare with and without government intervention for stochastic and determinate economies with various skill level pairs; $p = 1/3$, $l = 1.25$, $G = 0$.

VI. Determinate Economy with Diverse Skills

Paralleling the analysis above we can consider a determinate economy where individuals differ in both skill and length of working life. We can inquire into the relative importance of redistribution across these two dimensions. For ease of reference, we will inaccurately refer to redistribution across individuals with different lengths of working life as providing insurance. In Table 6 we describe the determinate economy under different tax regimes. Comparing Table 6 with Table 4, we see that the provision of information about length of working life raises social welfare by 2.7 percent in the absence of any government intervention. We also see that there are smaller gains to be had from tax interventions in the determinate economy. Table 5 reveals that optimal tax rates differ less between the stochastic and determinate economies when there is more *ex ante* inequality of skill in the population.

As above we have considered different partial optimization plans. To redistribute across skills without redistributing between those with different lengths of working life we have given different subsidies to those with different lengths of working life so that the budget is balanced within each group. If the tax is levied only in the first period the optimal rate is 8 percent. Allowing taxation in both periods gives optimal rates of 10 percent and 8 percent in the two periods.

A similar breakdown of the total gain is available for first-best instruments. Redistributing across skills but balancing the budget within each group of different working lives yields a social welfare of .1223. Redistributing across different lengths of working life but balancing the budget for each skill class yields a social welfare level of .1190.

	No Intervention	Second-best (Linear Taxation) Optima: Redistribution $t_2>0$	Second-best: Redistribution $t_2=0$	Second-best: Insurance	Second-best: Full	First-best Optima: Redistribution	First-best: Insurance	First-best: Full
average x_{1L}	(.225)	(.214)	(.220)	(.219)	(.209)	.225	(.209)	.209
average x_{1U}	(.167)	(.161)	(.162)	(.170)	(.164)	.167	(.209)	.209
average x_{2L}	(.281)	(.268)	(.275)	(.274)	(.262)	.281	(.261)	.261
average x_{2U}	(.208)	(.201)	(.202)	(.212)	(.205)	.208	(.261)	.261
average y_{1L}	.550	(.522)	(.521)	.562	(.534)	(.528)	.582	(.562)
average y_{1U}	.667	(.642)	(.647)	.661	(.633)	(.651)	.582	(.562)
average y_{2L}	.437	(.416)	(.449)	.404	(.390)	(.410)	.477	(.452)
output in period 1 per capita	.294	.282	.282	.297	.284	.294	.291	.291
output in period 2 per capita	.146	.139	.150	.135	.131	.146	.159	.239
present value of total output per capita	.411	.393	.402	.405	.389	.411	.418	.482
(t_1,t_2)	(0,0)	(.10,.08)	(.08,0)	(.00,.08)	(.10,.14)			
present value of income tax receipts per capita	0	.037	.023	.009	.043			
aggregate savings per capita	.089	.085	.081	.095	.090	.069	.082	.082
w	.1171	.1174	.1173	.1173	.1176	.1223	.1190	.1242

numbers in (parentheses) are averages; others are the same for all skill levels.

Table 6. Description of the determinate economy in which half of the individuals have $n = .393$ and the other half have $n = .607$, with $p = 1/3$, $I = 1.25$, $G = 0$.

These welfare levels lie in the interval of .1171 for no intervention and .1242 for full optimization. It is interesting that savings fall when the government intervenes, with the decrease being much larger when there is just redistribution. Comparing savings levels in Tables 4 and 6 one sees that individual uncertainty plays a major role in the determination of aggregate savings.

VII. Economies with a Continuum of Skills

To pursue comparative statics, we might allow both the level of skill and the expected length of working life to vary continuously in the population. We have not considered this second dimension of variation. Instead, we turn now to an approximation to a continuous skill distribution with an economy with 100 skill classes uniformly spread between 0 and 1.[1] Each individual continues to have a probability of 2/3 of being able to work in the second period.

In Figure 5 we show the social indifference contours in (t_1,t_2) space when $p = 1/3$, $G = 0$, and $I = 1.25$. Redistributive factors are strong enough in this population to have optimal taxes of .42 and .37, whereas a single consumer would only subject himself to 4 and 17 percent tax rates. This compares closely with the two-skill economy where the lower skill group has $n = .20$.

In Figure 6 we show the cross-section pattern of the expected present value of pretax earnings and consumption. (Expected utilities and equivalent uniform consumption levels are shown in Figures 7a and 7b.) In the absence of government intervention consumption and earnings coincide in lifetime terms as shown by the dotted straight line. In contrast, in the first-best economy, higher skilled individuals work more than lower skilled individuals, but pay larger lump-sum taxes. Thus while everyone enjoys the same consumption, those with higher skill levels have lower utility. In contrast to the economy without intervention, optimal linear taxes (which consist of a positive poll subsidy and positive income tax rates) discourage work -- with everyone having lower expected work. There

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[1]Our objective here is to understand the properties of this stochastic problem rather than to simulate the distribution of skills in the economy. First-best calculations are performed using a continuous distribution rather than a discrete approximation.

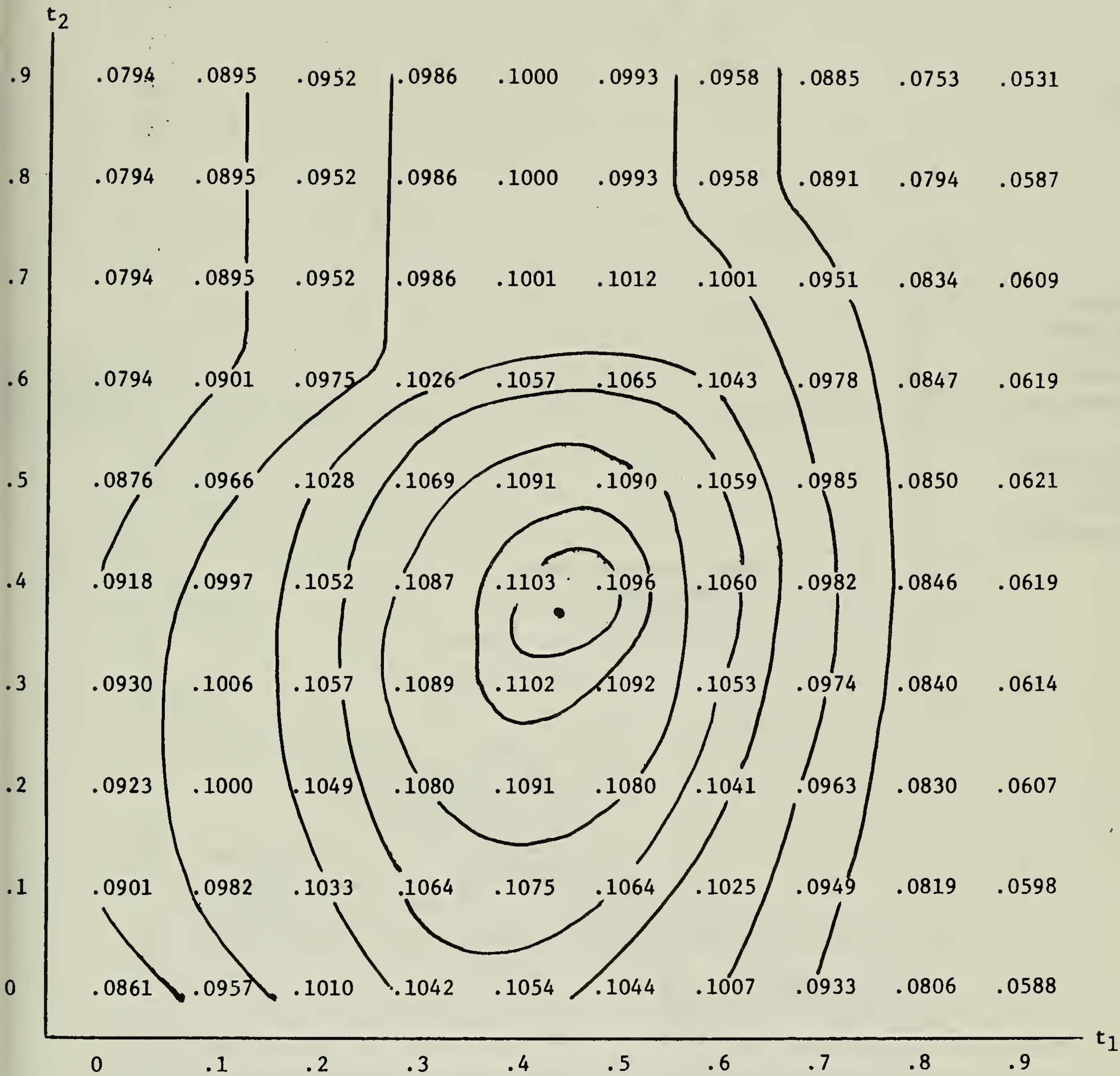

Figure 5. Social indifference curves in (t_1, t_2) space for the stochastic economy of individuals with uniformly distributed skill levels. Numbers given are social welfare levels for p = 1/3. I = 1.25, G = 0.

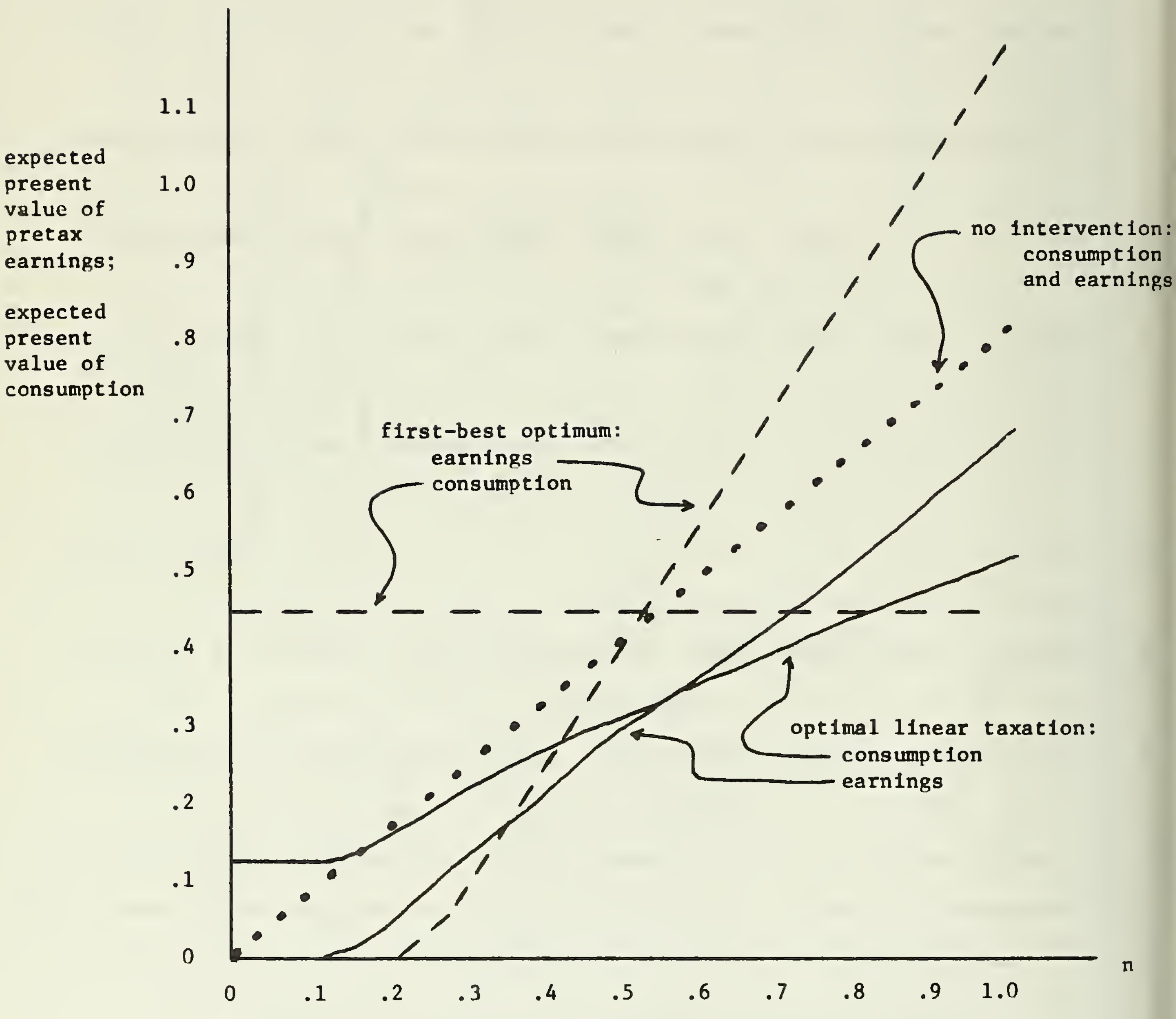

Figure 6. Consumption and earnings profiles without government intervention, with optimal linear taxation, and at the first-best optimum.

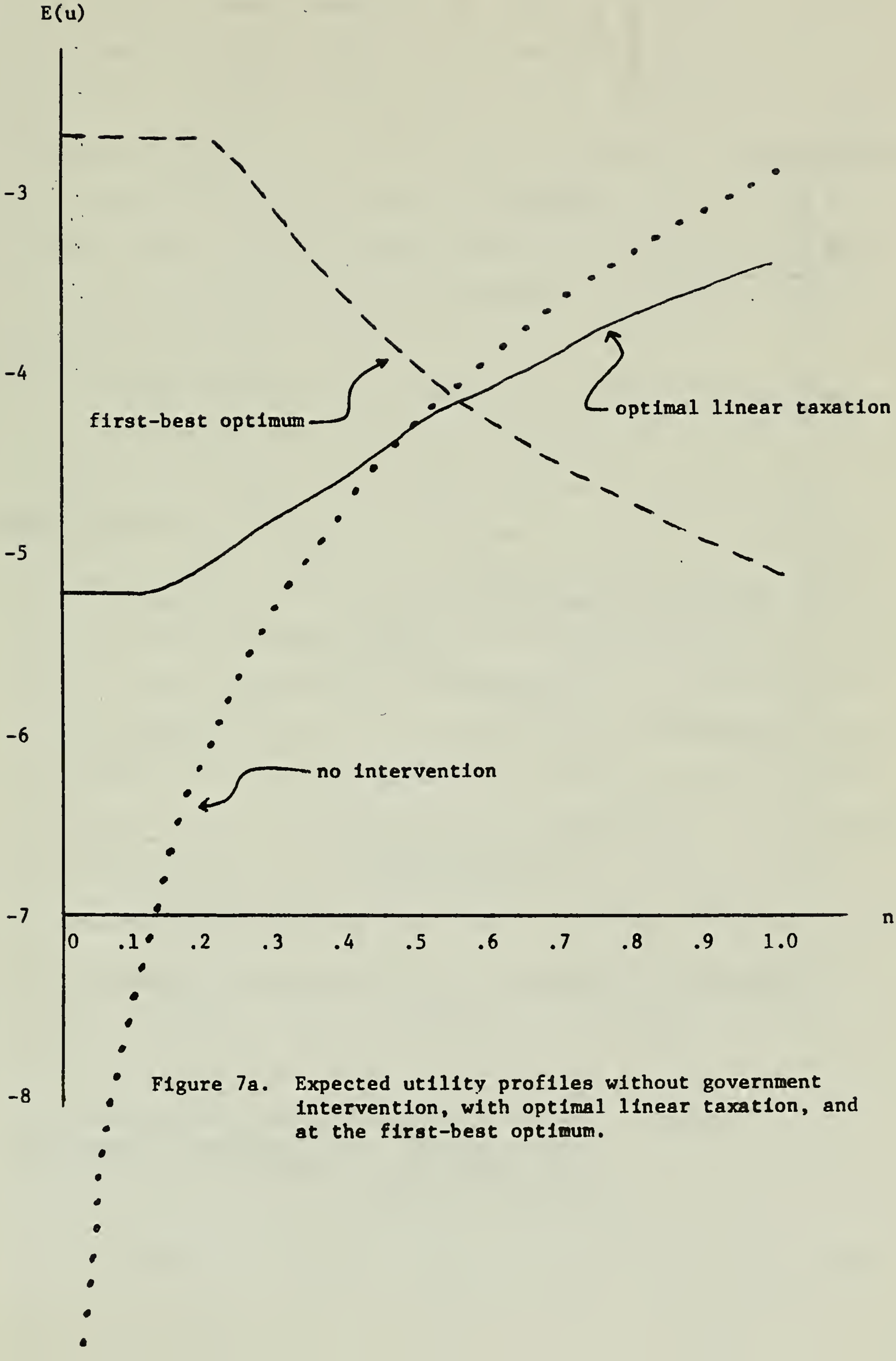

Figure 7a. Expected utility profiles without government intervention, with optimal linear taxation, and at the first-best optimum.

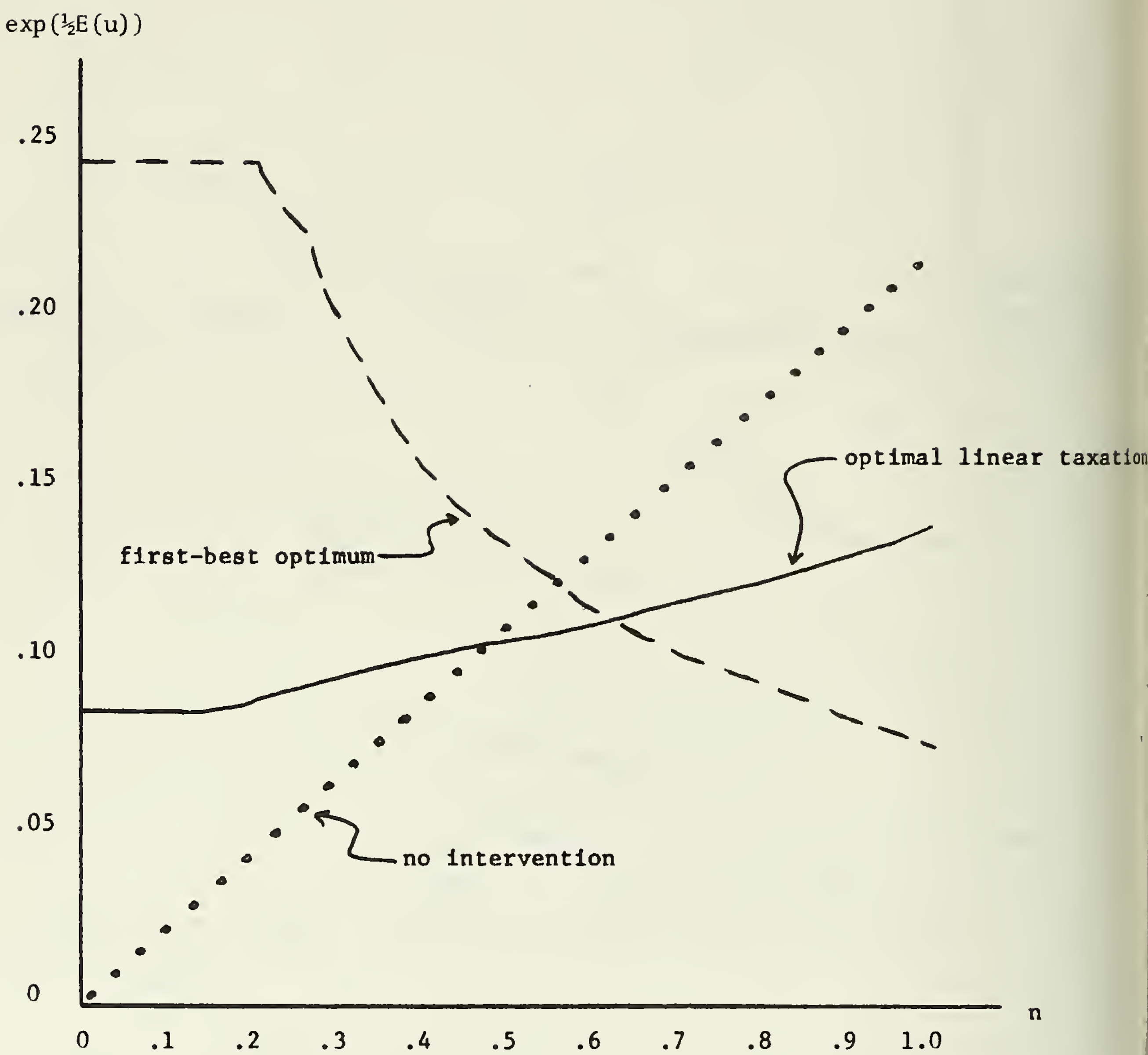

Figure 7b. Uniform consumption equivalent of expected utility profiles without government intervention, with optimal linear taxation, and at the first-best optimum.

is a fall in aggregate consumption and a shift in consumption toward those with lower skills; thus expected consumption goes up for the bottom 22 percent of the skill classes while it goes down for the remaining 78 percent. Since the diagram gives the expected value calculation, it does not present the insurance aspects of the change in consumption. This is shown in Figures 8 and 9. From these figures we see that the present discounted value of consumption increases for 33 percent of the unlucky and for 18 percent of the lucky.

As in the two-skill-class economy we can examine separately the insurance and redistributive potential in the economy. Consider an individual in the economy. He is making labor and consumption choices to maximize his utility given that he receives the subsidy and faces the tax rates that are optimal for the economy at large. Given his demands, his tax payments will exceed (fall short of) his poll subsidy by a certain amount -- which amount is equal to his contribution to (receipts from) the redistribution program of the economy at large. Now suppose instead that he were to regard this contribution as fixed and that he could then select any tax rates and subsidy level he desired subject to the constraint that his tax payments still exceed his chosen subsidy by the amount of the required contribution. This problem is of course equivalent to asking what tax system a single individual would subject himself to when the grant G is set equal to the receipts from the redistribution program. Since in the economy of identical individuals the optimal tax rates decline with G, we find here that those with higher (lower) skills would choose higher (lower) tax rates for insurance purposes only, holding constant the pattern of redistribution in the economy.

For example, we find that the individuals in the economy discussed above, facing income taxes of .42 and .37, receive a poll subsidy of .13.

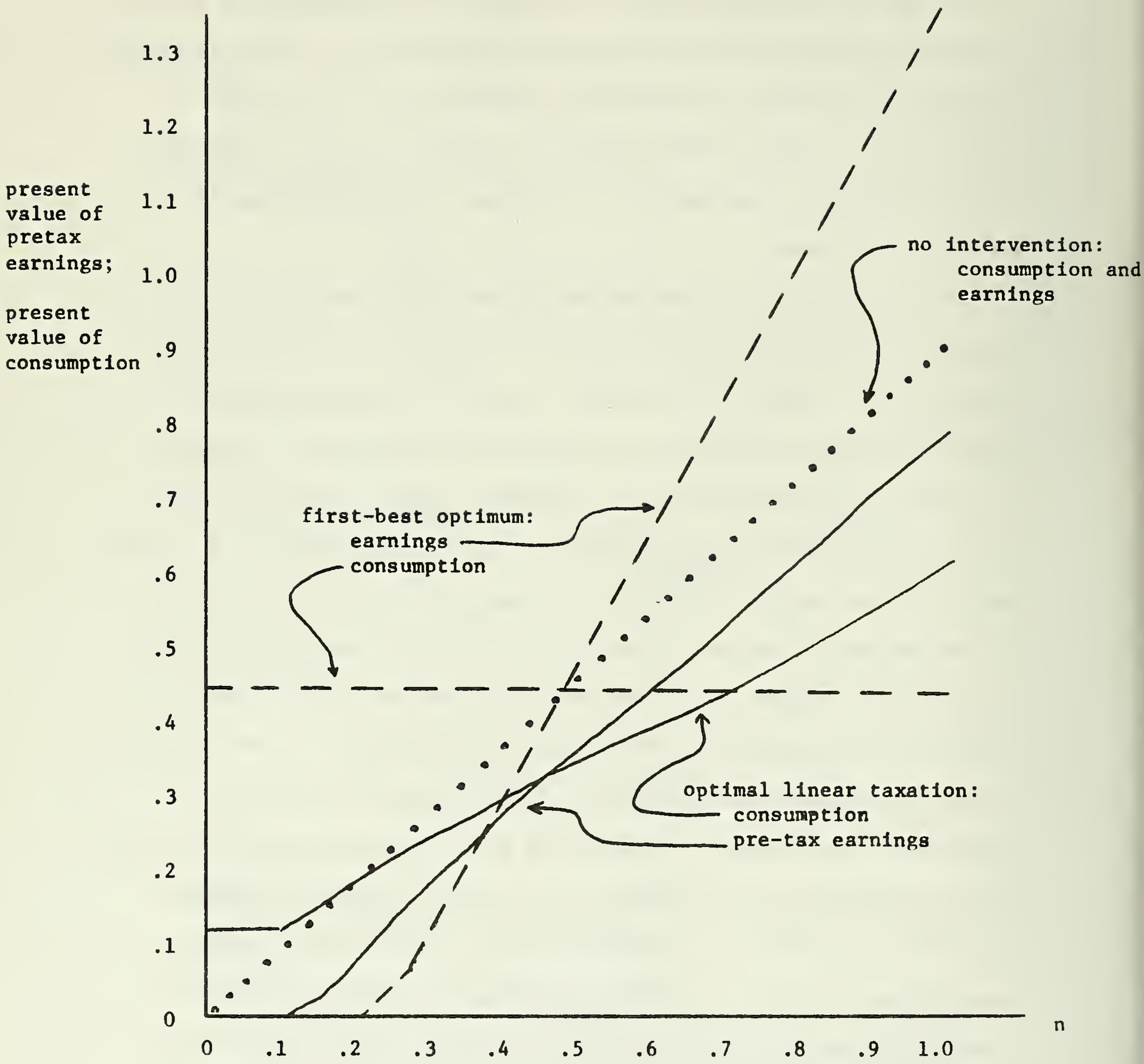

Figure 8. Consumption and earnings profiles without government intervention, with optimal linear taxation, and at the first-best optimum for lucky individuals.

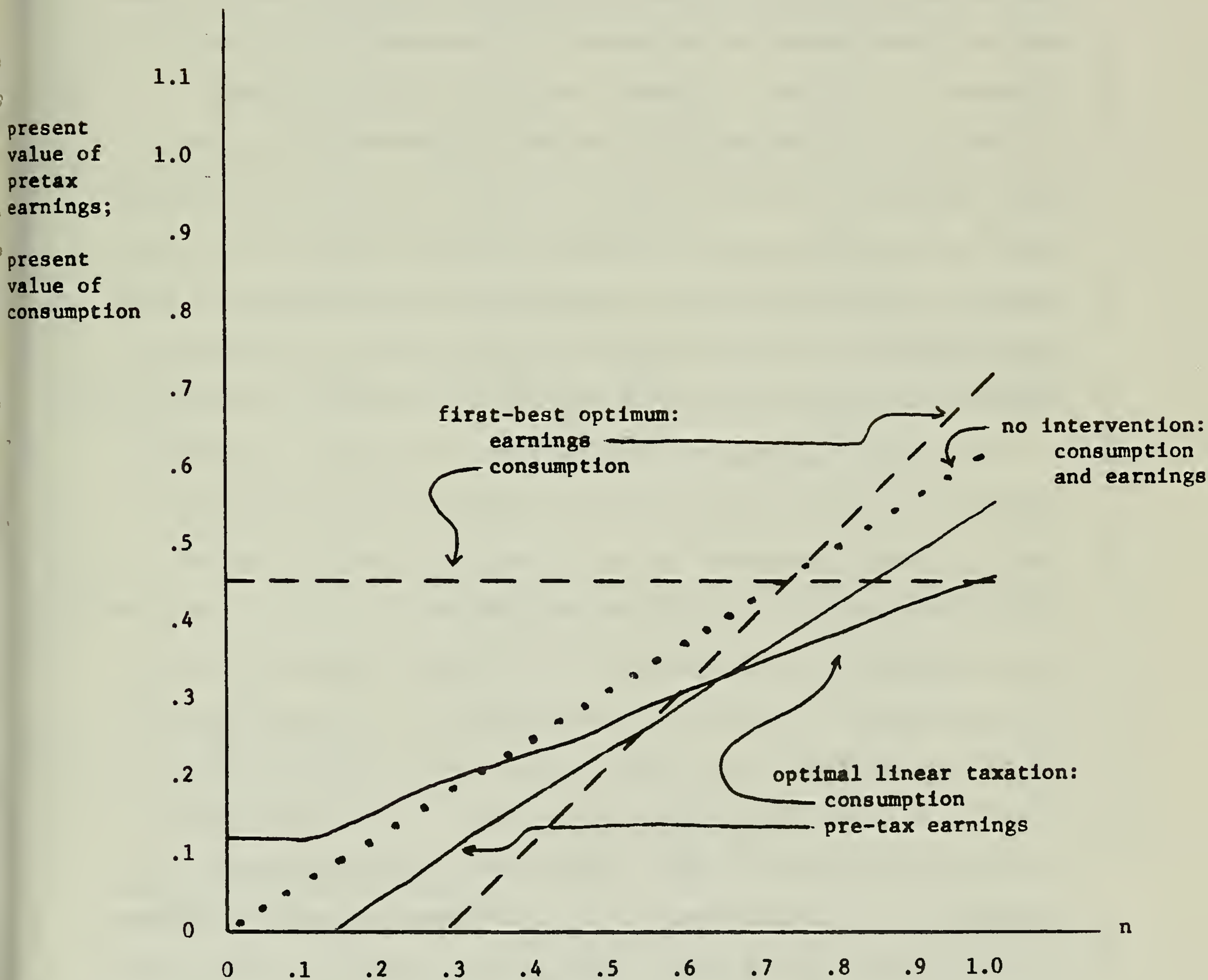

Figure 9. Consumption and earnings profiles without government intervention, with optimal linear taxation, and at the first-best optimum for unlucky individuals.

An individual whose marginal product of labor is .23 would prefer, however, to subject himself to tax rates of only .02 and .07 and to have his subsidy reduced to .10, which would result in his still receiving the same net transfer from the rest of the economy: .09. Without the deadweight burden from the redistributional system, his consumption and labor supply increase, and the uniform consumption equivalent of his expected utility ($\exp(\frac{1}{2}E(u))$) increases by 6.3 percent. Similarly, the person with n = .5, whose contribution to the redistribution program is approximately zero, would prefer to have his tax rates reduced to .04 and .17 which, as we have noted above, are the optimal taxes for a single individual when he is on a break-even basis. Thus we find that if redistribution were accomplished through lump-sum taxation rather than through distorting taxes, then individuals would want considerably reduced tax rates for the provision of insurance.

The relative importance of redistributional as opposed to insurance considerations in the economy can also be demonstrated by introducing these two elements separately via restricted linear taxation. We can provide insurance without redistribution by optimizing t_1 and t_2, returning to each individual as a subsidy exactly the present discounted value of his tax payments.[2] Because of the homogeneity properties mentioned above, the optimal tax rates are equivalent to those found for a single skill class. As has been noted, these rates are low -- .04 and .17 -- as is the improvement in social welfare. (See Table 7.) This improvement is equivalent, in the economy with a continuum of skill levels, to an increase in total output of 0.26 percent, which amounts to only 2.1 percent of the benefits available from optimal linear taxation.

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[2]Note that the poll subsidy will not be the same for individuals with different skill levels; it will be proportional to n.

	No Intervention	Second-best (Linear Taxation) Optima			First-best Optima		
		Redistribution	Insurance	Full	Redistribution	Insurance	Full
average x_{1L}	(.189)	(.154)	(.186)	(.153)	.198	(.209)	.221
average x_{1U}	(.189)	(.154)	(.186)	(.153)	.198	(.209)	.221
average x_{2L}	(.326)	(.307)	(.291)	(.232)	.359	(.261)	.277
average x_{2U}	(.152)	(.112)	(.167)	(.144)	.153	(.261)	.277
average y_{1L}	.622	(.405)	.612	(.395)	(.481)	.582	(.445)
average y_{1U}	.622	(.405)	.612	(.395)	(.481)	.582	(.445)
average y_{2L}	.348	(.345)	.300	(.222)	(.274)	.477	(.368)
output in period 1 per capita	.311	.243	.306	.242	.321	.291	.303
output in period 2 per capita	.116	.130	.100	.093	.137	.159	.174
present value of total output per capita	.404	.347	.386	.316	.431	.418	.443
fraction not working in period 1 if lucky	0	.09	0	.11	.198	0	.221
fraction not working in period 1 if unlucky	0	.09	0	.11	.198	0	.221
fraction not working in period 2 if lucky	0	.06	0	.14	.359	0	.277
(t_1, t_2)	(0,0)	(.41,0)	(.04,.17)	(.42,.37)			
present value of income tax receipts per capita	0	.100	.026	.129			
aggregate savings per capita	.122	.090	.120	.088	.123	.082	.081
w	.0861	.1055	.0866	.1105	.1356	.0899	.1425

numbers in (parentheses) are averages; others are the same for all skill levels.

Table 7. Description of the stochastic economy in which skill is uniformly distributed, with p = 1/3. I = 1.25, G = 0

On the other hand, we can introduce redistribution without insurance by not allowing transfers between states of nature; i.e., we can optimize t_1 subject to $t_2 = 0$. This yields a much larger tax ($t_1 = .4$) and much larger gains in social welfare (equivalent to a 9.8 percent increase in output). Thus, while this is a very restrictive way of redistributing without insuring, it enables us to gain 79.5 percent of the available welfare benefits of optimal linear taxation.[3] In the first-best setting, perfect insurance alone accounts for 6.7 percent of the welfare gains which are realized by introducing first-best insurance and redistribution together into the stochastic economy. On the other hand, redistribution by itself achieves 95.4 percent of the potential welfare gains in the first-best economy. And while optimal linear taxation without redistribution achieves only 14 percent of the available welfare gain from perfect insurance, optimal taxation without insurance captures 39 percent of the (much larger) potential gain from perfect redistribution. In combination, fully optimal linear taxation gives 43 percent of the welfare improvement which is obtained at the full first-best optimum.

Thus we find that the potential gains from providing insurance in the stochastic economy are small relative to the gains from perfect (nondistorting) redistribution. Moreover, linear taxation is much less effective in providing the achievable insurance protection than it is in redistributing income.

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[3] Of course, as the inequality in earning ability in the economy is decreased, the relative importance of distributional factors decreases as well. But the presence of any sizeable percentage of individuals with skill near zero does much to insure that distributional considerations will predominate. If, for example, the population is distributed uniformly except that 17 percent of the population is concentrated at $n = .005$, redistribution alone achieves 87.5 percent of the welfare gains from linear taxation while insurance alone accounts for a scant 0.5 percent of the possible improvement.

In Table 8 we repeat the analysis for the determinate analog to the stochastic economy with uniformly distributed skill levels. Then three separate social choice problems can again be distinguished: 1) we can find the full (second-best) optimum by maximizing welfare with respect to t_1, t_2, and s subject to the aggregate resource constraint; 2) we can insure without redistributing by returning to each consumer exactly the present discounted value of his total tax receipts as a lump-sum subsidy; and 3) we can redistribute without insuring by requiring that the aggregate resource constraint hold separately for the lucky and the unlucky.

The determinate economy has approximately the same optimal tax rates, .41 and .38, as are found for the stochastic economy. However, in the presence of uncertainty tax revenues are larger.[4] Welfare is again rather insensitive to changes in the tax rates. Welfare in the untaxed stochastic economy falls 38.4 percent short of that in the economy with first-best redistribution, which is equivalent to 27.2 percent of the output in the untaxed economy. Removing uncertainty reduces this gap slightly to an equivalent of 26.0 percent of output, a much smaller change from the provision of information than in the economy with a smaller spread in skills. Introducing taxation for redistributive purposes only into the determinate economy[5] leads to the realization of approximately the same percentage of the potential gains from first-best redistribution as in the case of the stochastic economy -- 32 percent as compared to 36 percent.

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[4]In the economy in which individuals have different marginal products in period one, as in the economy of identical consumers, there are some values of t_1 and t_2 for which social welfare is higher in the stochastic economy than in the economy which is identical but for the absence of uncertainty.

[5]To maintain comparability, we require that $t_2 = 0$ in the determinate, as well as in the stochastic economy.

	No Intervention	Second-best (Linear Taxation) Optima				First-best Optima		
		Redistribution		In-surance	Full	Redis-tribution	In-surance	Full
		$t_2>0$	$t_2=0$					
average x_{1L}	(.225)	(.173)	(.199)	(.219)	(.171)	.241	(.209)	.221
average x_{1U}	(.167)	(.137)	(.142)	(.170)	(.143)	.172	(.209)	.221
average x_{2L}	(.281)	(.216)	(.248)	(.274)	(.213)	.302	(.261)	.277
average x_{2U}	(.208)	(.171)	(.178)	(.212)	(.177)	.214	(.261)	.277
average y_{1L}	.550	(.349)	(.342)	.562	(.362)	(.416)	.582	(.445)
average y_{1U}	.667	(.454)	(.484)	.661	(.417)	(.526)	.582	(.445)
average y_{2L}	.437	(.260)	(.456)	.404	(.261)	(.337)	.477	(.368)
output in period 1 per capita	.294	.233	.225	.297	.233	.306	.291	.303
output in period 2 per capita	.146	.111	.168	.135	.111	.163	.159	.174
present value of total output per capita	.411	.322	.360	.405	.322	.436	.418	.443
fraction not working in period 1 if lucky	0	.12	.06	0	.11	.241	0	.221
fraction not working in period 1 if unlucky	0	.09	.08	0	.11	.172	0	.221
fraction not working in period two if lucky	0	.15	.04	0	.14	.302	0	.277
(t_1,t_2)	0	(.41,.37)	(.35,0)	(.00,.08)	(.41,.38)			
present value of income tax receipts per capita	0	.128	.079	.009	.129			
aggregate savings per capita	.089	.072	.045	.095	.072	.088	.082	.082
w	.0885	.1115	.1052	.0886	.1121	.1399	.0899	.1425

numbers in (parentheses) are averages; others are the same for all skill levels

Table 8. Description of the determinate economy in which skill is uniformly distributed, with p = 1/3. I = 1.25, G = 0

For many of the policy configurations savings are much higher when there is individual uncertainty. Thus, while the impact of uncertainty is small relative to distributional considerations, it does play a significant role in the optimization problem, even if insurance is not to be provided at all.

VIII. Results for a More Risk Averse Consumer

The Cobb-Douglas model, being intertemporally additive, shows no particular gain from giving consumption to those who have been accustomed to high consumption. But maintaining income during retirement at some appropriate fraction of preretirement earnings is a common goal of social insurance systems. To see the effects of incorporation of this idea into the specification of the utility function, we consider a polar case of interdependence between the utility of consumption in the two periods.

Suppose that consumers desire consumption in the two periods to be in some exact ratio, and that consumption in one period which is in excess of the level that would be desired given consumption in the other period provides no added utility. We define expected utility to be

$$u = 2(1-p)\ \ln(\min(x_1,\ \theta x_{2L})) + 2p\ \ln(\min(x_1,\ \theta x_{2U})) + \ln(1-y_1) + (1-p)\ \ln(1-y_{2L}) - \ln(\theta) \tag{13}$$

where θ is the inverse of the desired "replacement ratio." Geometrically, indifference curves between first- and second-period consumption are right angles. To make our results comparable to those discussed above for the Cobb-Douglas utility function, we will take as a value of θ the value I^{-1}. Then individual behavior under certainty and the first-best optimum will be the same as those described above for the Cobb-Douglas utility function.

First, we ask how effectively we can insure such a consumer against loss of skill in the second period. Optimal taxes for the provision of insurance alone are even smaller than in the Cobb-Douglas case -- .01 and .08 -- and only 0.5 percent of the welfare gain that would result from perfect insurance is realized. This is shown in Table 9. The same risk aversion that makes insurance so desirable (the gains from first-best insurance are 2.7 times as great in the present case as they are when consumers have Cobb-Douglas utility functions) leads to a work disincentive problem in the second period which almost totally precludes the consumer from being insurable. In contrast, recall that optimal linear taxes give the Cobb-Douglas consumer 13.7 percent of the gain in social welfare that would come from first-best insurance.

The role of uncertainty in affecting both savings and second-period labor supply is very marked. In the absence of government intervention, second-period output is only 3.9 percent of first-period output. In the analogous determinate economy the ratio is 50 percent. Savings in the uncertain economy is 78 percent higher than in the determinate analog. The introduction of optimal linear taxes practically eliminates second-period output and results in a small increase in savings. Moving to the first-best economy nearly halves aggregate savings and raises second-period output to approximately half of first-period output.

Turning to the economy of consumers with uniformly distributed skill levels, we again consider the benefits from pure insurance and from pure redistribution. In Table 10 we show the calculations for the economy in which consumers have the utility function which we have been considering. Fully optimal taxes in the economy are large -- .43 and .23 -- but even more so than in the Cobb-Douglas case it is redistribution that is responsible for the high tax rates as well as for the gains in social welfare. Pure insurance yields only 0.2 percent of the gains from optimal linear taxation, whereas pure redistribution yields 96.4 percent of the welfare gains. (For the Cobb-Douglas consumer these gains were 2.1 percent

	No Government Intervention	Optimal Linear Taxes	First-best Optimum
x_1	.174	.170	.209
x_{2L}	.217	.213	.261
x_{2U}	.198	.202	.261
y_1	.664	.662	.582
y_{2L}	.038	.023	.477
Output per capita in period one	.332	.331	.291
Output per capita in period two	.013	.008	.159
Present value of total output	.342	.337	.418
Aggregate first-period savings per capita	.158	.161	.082
w	.1077	.1078	.1218

Table 9. Description of the stochastic economy of identical individuals with the utility function described in this section, with p = 1/3, n = .5, I = 1.25, and G = 0.

	No Intervention	Second-best (linear Taxation) Optima			First-best Optima		
numbers in (parentheses) are averages; others are the same for all skill levels.		Redistribution	Insurance	Full	cRedistribution	Insurance	Full
average x_{1L}	(.174)	(.147)	(.170)	(.140)		(.209)	.221
average x_{2U}	(.174)	(.147)	(.170)	(.140)		(.209)	.221
average x_{2L}	(.217)	(.184)	(.213)	(.175)		(.261)	.277
average x_{2U}	(.198)	(.151)	(.202)	(.161)		(.261)	.277
average y_1	.664	(.445)	.662	(.440)		.582	(.445)
average y_{2L}	.038	(.050)	.023	(.026)		.477	(.368)
output in period 1 per capita	.332	.268	.331	.267		.291	.303
output in period 2 per capita	.013	.022	.008	.013		.159	.174
present value of total output per capita	.342	.286	.337	.277		.418	.443
fraction not working in period 1 if lucky	0	.10	0	.10		0	.221
fraction not working in period 1 if unlucky	0	.10	0	.10		0	.221
fraction not working in period 2 if lucky	0	.18	0	.33		0	.277
(t_1, t_2)	(0,0)	(.42,0)	(.01,.08)	(.43,.23)			
present value of income tax per capita	0	.113	.004	.117			
aggregate savings per capita	.158	.121	.161	.126		.082	.081
w	.0795	.1001	.0796	.1008		.0899	.1425

Table 10. Description of the stochastic economy in which skill is uniformly distributed and individuals maximize the utility function described in this section, with $p = 1/3$, $I = 1.25$, $G = 0$.

and 79.5 percent, respectively.) With optimal linear taxes, 10 percent of the consumers do not work at all and an additional 23 percent of the lucky do not work in period two. (In the Cobb-Douglas economy these numbers are 11 percent and 3 percent.) And expected output in period two represents only 5 percent of the present value of expected total output (as compared to 29 percent in the Cobb-Douglas economy). Despite the greater need for insurance here than in the Cobb-Douglas case, second-period taxes are much lower due to the moral hazard problem. (When $t_1 = .4$ and t_2 is greater than .41, there is no production at all in period two.)

Thus for a consumer who is extremely risk averse in this sense, the benefits from first-best insurance are large, the benefits realized from second-best insurance are small, and the moral hazard problem is highly significant.

IX. Pensions with an Earnings Test

The linear tax system has been found to be surprisingly ineffective in providing the consumer with insurance against the contingency of loss of earning ability in the second period. We therefore are led to examine how much more effectively simple nonlinear taxation can transfer income between states of nature. We shall consider two ways of introducing nonlinearity into second-period income taxation. One way parallels negative income taxation for the elderly or a universal pension with an earnings test by allowing a higher marginal tax rate for low incomes. Secondly we will parallel social security by giving a second-period benefit which depends (linearly) on first-period earnings. This benefit is reduced linearly with second-period earnings. Thus, again, we will have a larger marginal tax rate on low than on high earnings, with the break point determined by the size of social security benefits and the rate of decline of benefits with earnings. Since either this universal pension system or this wage-related pension system add two more parameters into the calculation, we have not attempted to consider both at once in the calculations, although we present the model with both systems present.[1]

Suppose a payroll tax is levied on earnings (and is merged with the income tax in this model), and benefits, paid in the second period, consist of a minimum benefit of a plus Ib dollars for every dollar of earnings in period one. In addition, there may be a retirement test and c dollars of benefits are withheld for each dollar of second-period

[1] Because social welfare is not concave in the tax parameters, it would have been prohibitively expensive to search the six-dimensional (t_1, t_2, s, a, b, c) space to find the full optimum.

earnings. Total benefits are thus the bracketed amounts in the revised consumer budget constraints:

$$\begin{aligned} x_{2U} &= I(s + (1-t_1)ny_1 - x_1) \\ &\quad + [a + bIny_1] \\ x_{2L} &= I(s + (1-t_1)ny_1 - x_1) + (1-t_2)ny_{2L} \\ &\quad + [a + bIny_1 - cny_{2L}]_+. \end{aligned} \tag{14}$$

where $[\;]_+$ refers to the maximum of the amount in brackets and zero. Note that benefits are constrained to be nonnegative. Thus an unlucky person consumes accumulated savings plus the social dividend plus benefits from the pension system. A lucky person consumes this plus earnings net of both the income tax and the offset in benefits for second-period earnings.

The government budget constraint becomes, for the economy of identical individuals,

$$\begin{aligned} & Is + (1-p)[a + bIny_1 - cny_{2L}]_+ + p(a + bIny_1) \\ & = It_1y_1 + (1-p)t_2y_{2L} + IG. \end{aligned} \tag{15}$$

That is, the social dividend plus pension benefits to the lucky and the unlucky equal income tax revenue plus the grant provided from outside of the system.

We now consider separately the two ways to introduce nonlinear taxation. We start with the economy of identical consumers. There are two aspects of labor supply response in the second period. The individual can respond to a small change in taxes by a small change in work or by ceasing work altogether. The latter constraint will be our sole concern when consumers are identical. To analyze this in isolation we will use a zero marginal tax rate on second-period work. Thus we set t_2 equal to zero and c equal to one. This simplification will not continue once there are diverse skills in the economy. With the single consumer, the optimal universal pension is an optimal pair (a,t_1), while the optimal wage-related pension is an optimal pair (b,t_1).

With a universal pension, the budget constraint is $It_1ny_1 = pa$, and so the consumer is receiving the expected value of his tax payments as a subsidy if he is unable to work. With a wage-related pension, $t_1 = pb$. In either case, the conditional subsidy (a or $Ibny_1$) cannot be made any larger than the value which would leave the worker indifferent to continuing work when lucky. Nevertheless the two approaches are not equivalent. If the pension which is perceived as a lump sum is replaced by a plan in which the pension depends on the individual's earning record, the worker perceives a further gain from working in period one than was seen before. However, this additional work (and additional savings) will undercut the scheme because it would lead him to stop working when lucky. Thus the maximal scheme must have a smaller level of b than would give the same pension at the level of work previously chosen. With a wage-related pension the individual perceives a zero marginal expected tax on first-period work. This avoidance of marginal distortions, however, is achieved at the cost of being constrained to a smaller pension. Thus it is not clear which method of insurance is preferable. We calculated the optimal parameter values for the two schemes in an economy with $n = .5$, $I = 1.25$, $p = 1/3$, and $G = 0$. The results are shown in Table 11. The earnings-related social security program yields welfare gains which amount to 45 percent of the potential gains from first-best insurance. The universal

	No Government Intervention	Optimal Linear Taxes	Wage-Related Pension	Flat Pension	First-Best Optimum
x_1	.189	.186	.198	.192	.209
x_{2L}	.326	.291	.308	.300	.261
x_{2U}	.152	.167	.177	.171	.261
y_1	.622	.612	.613	.582	.582
y_{2L}	.348	.299	.384	.399	.477
output in period one per capita	.311	.306	.307	.291	.291
output in period two per capita	.116	.100	.128	.133	.159
present value of total output per capita	.404	.386	.409	.398	.418
aggregate first-period savings per capita	.122	.120	.109	.099	.082
t_1	--	.04	.053	.082	
t_2	--	.17	--	--	
a	--	--	--	.070	
b	--	--	.159	--	
pension	--	--	.061	.070	
w	.1167	.1174	.1190	.1192	.1218

Table 11. Introducing Social Security into a Stochastic Economy of identical individuals with $p = 1/3$, $n = .5$, $I = 1.25$, $G = 0$.

pension plan fares slightly better, achieving 49 percent of the potential welfare gains. Thus in our example avoiding marginal distortions from taxation is less important than providing a larger conditional subsidy. Note that either program is a significant improvement over the performance of the linear tax system, which was able to capture only 13.7 percent of the potential insurance gains. The difference between the social welfare under linear taxation and under either of these public pension schemes is equivalent to approximately 1 percent of total output.

In considering the introduction of a public pension plan into the economy characterized by a diversity of skill levels, we will ask two questions. First, what gains in welfare can be realized by adding these programs to the optimal linear tax system; and second, how should the underlying tax system be modified to maximize the gains from the additional insurance systems.

We take as our starting point the economy in which the skill parameter n is uniformly distributed, $p = 1/3$, $I = 1.25$, and $G = 0$. The optimal linear taxes are .42 and .37. If we provide a flat pension benefit a of .015 (equal to 3.8 percent of per capita output) and subject it to an earnings test which reduces the benefit by 15 cents for every dollar of second-period earnings (the optimal (a,c) pair given that t_1 and t_2 are maintained at .42 and .37), social welfare increases by an amount which is equivalent to giving the government a grant equal to 0.32 percent of total output. (See Table 12.) Note that the price which must be paid for giving this subsidy to the poor (both those with low earning abilities and those who are unlucky) is the additional deadweight burden of inducing an additional 9 percent of those with second-period skills to leave the labor force. However, the upper 55 percent of the skill

	Optimal Linear Taxation $(t_1,t_2)^*$	flat pension (a > 0)			Social Security (b > 0)		
		Adding pension to System with $(t_1,t_2)^*$	Decreasing t_2 to level which is optimal given pension	Optimal pension, t_2 given t_1^*	Adding Soc. Sec. to System with $(t_1,t_2)^*$	Decreasing t_2 to level which is optimal given Soc. Sec.	Optimal Soc. Sec., t_2 given t_1^*
average x_1	.153	.153	.155	.155	.155	.154	.157
average x_{2L}	.232	.225	.262	.238	.228	.242	.236
average x_{2U}	.145	.151	.130	.147	.152	.140	.149
average y_1	.395	.398	.391	.394	.400	.392	.399
average y_{2L}	.222	.192	.284	.217	.211	.245	.234
output in period 1 per capita	.242	.243	.239	.240	.242	.240	.240
output in period 2 per capita	.093	.089	.113	.101	.094	.100	.102
present value of total output per capita	.316	.313	.329	.320	.317	.321	.322
fraction not working in either period	.11	.11	.11	.11	.11	.11	.10
fraction working in period 1 but not working in period 2 if able	.03	.12	.0	.08	.18	.01	.17
fraction working in both periods if able, receiving positive pension benefits if lucky	--	.22	--	.44	--	--	--
fraction working in both periods if able, receiving no pension benefits if lucky	.86	.55	.89	.37	.71	.88	.73
(t_1,t_2,a,b,c)	(.42,.37, 0,0,0)	(.42,.37, .015,0,.15)	(.42,.22, 0,0,0)	(.42,.22, .05,0,.25)	(.435,.37, 0,.045,.2)**	(.42,.32, 0,0,0)	(.44,.32, 0,.06,.63)**
present value of total tax receipts per capita	.129	.121	.120	.093	.129	.127	.127
aggregate savings per capita	.088	.090	.084	.085	.087	.086	.084
w	.1105	.1108	.1094	.1111	.1110	.1103	.1111

Table 12. Introducing social security into the stochastic economy in which skill is uniformly distributed, with p = 1/3, I = 1.25, and G = 0.

distribution -- those who receive no benefits if lucky -- are barely affected by the decrease in s to finance the system. We find that the magnitude of c is not nearly as important as having the benefits confined to the poor. Indeed, raising c all the way to 1 sacrifices less than 7 percent of the gains from the introduction of the program.

Reducing the second-period tax rate t_2 increases the gains which may be achieved by decreasing the tendency to leave the labor force in period two. As is shown in Table 12, reducing t_2 to .22 allows the optimal negative income tax to yield benefits equal to an increase in output of 0.56 percent. This is 75 percent larger than can be achieved by introduction of a flat pension with no change in tax rates.

We find a similar pattern when we introduce a social security system. Returning to the linear tax optimum of $t_1 = .42$ and $t_2 = .37$, we choose social security benefits and the payroll tax by searching over b, c, and t_1 with $pb = (t_1 - .42)$. That is, we introduce a first-period payroll tax which would just finance the pension system if all the lucky worked. The rest of the financing comes from a decline in s. At the optimum, the welfare gain is equal to a 0.43 percent increase in output. This gain can be increased to 0.50 percent by lowering t_2 to .32. In both cases, at the optimal c no one who works in period two receives a social security benefit. There is a wide range of values of c that accomplish this end.

Thus we find that while social insurance and the negative income tax can increase welfare by a nonnegligible amount, reduction of the tax rate in period two is called for if benefits are to be maximized.

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